EXERCISES FROM
A FRIENDLY INTRODUCTION TO GROUP THEORY
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Contents

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1. Preliminaries

1.1. Sets.

1. (a)

```
IsUnder30 := n -> n < 30;;


# {n ∈ ℤ | n² < 30}
S := [];; n := 0;;
repeat
    Add( S, n^2 );
    n := n + 1;
until not IsUnder30( n^2 );;


# {x, y, z ∈ S | x² + y² + z²}
T := Set( List( Tuples( S, 3), Sum) );;


# {n ∈ ℤ | n < 30 ∧ ∃ x, y, z ∈ ℤ, x² + y² + z² = n}
Answer := Set( Filtered( T, IsUnder30 ) );;
```

```
[ 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18,
  19, 20, 21, 22, 24, 25, 26, 27, 29 ]
```

(b)
```
DigitsInt := function ( n, base )
    local digits;
    digits := [];
    repeat
        Add( digits, RemInt( n, base ), 1 );
        n := QuoInt( n, base );
    until IsZero( n );
    return digits;
end;;


IsAllOddSortedList := function ( digits )
    return ForAll( digits, IsOddInt ) and
           IsSortedList( digits );
end;;


IsAllOddSortedDigitsInt := function ( n )
    return IsAllOddSortedList( DigitsInt( n, 10 ) );
end;;


Answer := Set( Filtered( [100 .. 999],
                         IsAllOddSortedDigitsInt ) );;
```

```
[ 111, 113, 115, 117, 119, 133, 135, 137, 139, 155, 157, 159,
  177, 179, 199, 333, 335, 337, 339, 355, 357, 359, 377, 379,
  399, 555, 557, 559, 577, 579, 599, 777, 779, 799, 999 ]
```

(c)
```
Answer := Cartesian( [ 1, 2, 3 ], [ 1, FLOAT.PI ] );;
```

```
[ [ 1, 1 ], [ 1, 3.141592653589793 ], [ 2, 1 ],
  [ 2, 3.141592653589793 ], [ 3, 1 ],
  [ 3, 3.141592653589793 ] ]
```

2. *Proof of* (**2**).

$$\dfrac{\dfrac{\dfrac{\overline{x \in R \cup (S \cap T)} \quad x \in R}{x \in (R \cup S) \cap (R \cup T)} \quad \dfrac{\overline{x \in R \cup (S \cap T)} \quad x \in S \cap T}{x \in (R \cup S) \cap (R \cup T)}}{R \cup (S \cap T) \subseteq (R \cup S) \cap (R \cup T)} \qquad \dfrac{\dfrac{\overline{x \in R \cup (S \cap T)}}{x \in (R \cup S) \cap (R \cup T)}}{R \cup (S \cap T) \subseteq (R \cup S) \cap (R \cup T)}}{R \cup (S \cap T) = (R \cup S) \cap (R \cup T)}$$

□

Proof of (**4**).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{x \in R \setminus (S \cap T)} \quad x \notin S}{x \in R \setminus S} \quad \dfrac{\overline{x \in R \setminus (S \cap T)} \quad x \notin T}{x \in R \setminus T}}{x \in (R \setminus S) \cup (R \setminus T)}}{R \setminus (S \cap T) \subseteq (R \setminus S) \cup (R \setminus T)} \qquad \dfrac{\dfrac{\dfrac{\overline{x \in (R \setminus S) \cup (R \setminus T)} \quad x \in R \setminus S}{x \in R \ \wedge \ x \notin S} \quad \dfrac{\overline{x \in (R \setminus S) \cup (R \setminus T)} \quad x \in R \setminus T}{x \in R \ \wedge \ x \notin T}}{x \in R \setminus (S \cap T)}}{(R \setminus S) \cup (R \setminus T) \subseteq R \setminus (S \cap T)}}{R \setminus (S \cap T) = (R \setminus S) \cup (R \setminus T)}$$

□

3. (a) *Proof.*

$$
\begin{array}{c}
\begin{array}{c}
x \in (A \cup B) \cup C \\ \hline
x \in A \cup B \vee x \in C \\ \hline
x \in A \vee x \in B \vee x \in C \\ \hline
x \in A \vee x \in B \cup C \\ \hline
x \in A \cup (B \cup C) \\ \hline
(A \cup B) \cup C \subseteq A \cup (B \cup C)
\end{array}
\quad
\begin{array}{c}
x \in A \cup (B \cup C) \\ \hline
x \in A \vee x \in (B \cup C) \\ \hline
x \in A \vee x \in B \vee x \in C \\ \hline
x \in A \cup B \vee x \in C \\ \hline
x \in (A \cup B) \cup C \\ \hline
A \cup (B \cup C) \subseteq (A \cup B) \cup C
\end{array}
\\ \hline
(A \cup B) \cup C = A \cup (B \cup C)
\end{array}
$$

□

(b) *Proof.*

$$
\begin{array}{c}
\begin{array}{c}
x \in (A \cap B) \cap C \\ \hline
x \in A \cap B \wedge x \in C \\ \hline
x \in A \wedge x \in B \wedge x \in C \\ \hline
x \in A \wedge x \in B \cap C \\ \hline
x \in A \cap (B \cap C) \\ \hline
(A \cap B) \cap C \subseteq A \cap (B \cap C)
\end{array}
\quad
\begin{array}{c}
x \in A \cap (B \cap C) \\ \hline
x \in A \wedge x \in (B \cap C) \\ \hline
x \in A \wedge x \in B \wedge x \in C \\ \hline
x \in A \cap B \wedge x \in C \\ \hline
x \in (A \cap B) \cap C \\ \hline
A \cap (B \cap C) \subseteq (A \cap B) \cap C
\end{array}
\\ \hline
(A \cap B) \cap C = A \cap (B \cap C)
\end{array}
$$

□

(c) *Proof.*

$$
\begin{array}{c}
\begin{array}{c}
x \in A \setminus (A \setminus B) \\ \hline
x \in A \wedge x \notin (A \setminus B) \\ \hline
x \in A \wedge (x \notin A \vee x \in B) \\ \hline
x \in A \wedge x \in B \\ \hline
x \in A \cap B \\ \hline
A \setminus (A \setminus B) \subseteq A \cap B
\end{array}
\quad
\begin{array}{c}
x \in A \cap B \\ \hline
x \in A \wedge x \in B \\ \hline
x \in A \wedge x \notin A \setminus B \\ \hline
x \in A \setminus (A \setminus B) \\ \hline
A \cap B \subseteq A \setminus (A \setminus B)
\end{array}
\\ \hline
A \setminus (A \setminus B) = A \cap B
\end{array}
$$

□